

BRIEF COMMUNICATION

MOTION OF A VAPORIZING DROPLET IN A CONSTANT CROSS FLOW

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1. INTRODUCTION

In many forms of chemical power plants the fuel is usually introduced into the reactor in the form of a spray. The subsequent processes of droplet vaporization, fuel (in both liquid and gas phases)—air mixing, and combustion usually take place in confined spaces. Hence an analysis of the motion of the droplets, resulting from their injection inertia and the subsequent interaction with aerodynamic forces, is essential to understanding the wall impingement of droplets, as well as the degree of mixedness between the droplets and the gas during this initial mode of coarse mixing.

Miesse (1954) analyzed the one-dimensional motion of a droplet by using the so-called D^2 -law (see, for example, Williams 1965) for the vaporization rate and Stokes law for the drag force. Waldman & Reinecke (1971), and Jaffe (1973), used a constant drag-coefficient, C_D , in analyzing droplet dynamics in hypersonic shock layers; hence implicitly assuming that the droplet Reynolds number, Re , is very large. In many practical situations, however, Re is neither so small that Stokes law is rigorously applicable, nor can it be very large without the droplet becoming unstable and disintegrating. Furthermore, in the presence of convective gas motion, the droplet vaporization rate is expected to exceed the value given by the D^2 -law, which assumes a stagnant atmosphere for the vaporization process.

These non-ideal behaviors were realistically accounted for in a recent theory by Law (1975a) for spray vaporization in one-dimensional flows. A similar analysis, with a refined drag-coefficient expression, is extended here to obtain closed-form solutions for the motion of a single droplet undergoing evaporation, with and without gas-phase reactions, in a constant cross flow.

2. GOVERNING EQUATIONS

The problem under consideration is the motion resulting from injecting a droplet with initial diameter D_o , velocity u_{po} and v_{po} in the x - and y -directions, respectively, into a gas stream with constant u_G in the x -direction. It is also assumed that the evaporation and motion of the droplet do not affect any of the properties of the gas stream.

The rate of decrease of the droplet size under forced convection is given by Williams (1965),

$$d\hat{D}/dt = -2kH(Re, Sc) \quad [1]$$

where

$$H(Re, Sc) = 1 + 0.276Sc^{1/3}Re^{1/2} \quad [2]$$

is the Frossling correction term (Williams 1965) that accounts for the increase in evaporation rate due to forced convection, $Sc = \mu_G/(\rho_G\delta_G)$ is the Schmidt number and

$$Re = [D_o\rho_G/\mu_G][(u_G - u_p)^2 + v_p^2]^{1/2}\hat{D} \quad [3]$$

is the droplet Reynolds number. Furthermore $\hat{D} = D/D_o$, $k = 4M/(D_o^2 \rho_p)$, $M = \rho_G \delta_G \ln(1+B)$ is a measure of the evaporation rate, and the transfer number B is $(C_G/L)(T_G - T_p)$ or $[iY_{OG}Q + C_G(T_G - T_p)]/L$ for vaporization without or with gas-phase reactions, respectively (Williams 1965). In the above t is the time, ρ is the density, T is the temperature, μ is the viscosity coefficient, δ is the binary-diffusion coefficient, C is the specific heat, i is the stoichiometric fuel-oxidizer ratio, Y_{OG} is the oxidizer mass fraction in the gas stream, L is the specific latent heat of vaporization, Q is the chemical heat release per unit mass of fuel consumed; and the subscripts o , p , and G respectively designate the initial state, the droplet, and the gas.

The accelerations the droplet experiences in the x - and y -directions are respectively

$$du_p/dt = (3\mu_G/4\rho_p)(C_D Re/D^2)(u_G - u_p), \quad [4]$$

and

$$dv_p/dt = -(3\mu_G/4\rho_p)(C_D Re/D^2)v_p, \quad [5]$$

where the drag coefficient C_D is given by the form proposed by Law (1975a)

$$C_d = K(Sc) \times H(Re, Sc) \times G(B)/Re. \quad [6]$$

In [6] the function $G(B)$ accounts for the change in drag due to the outward mass-transfer at the droplet surface. Various expressions for $G(B)$ have been suggested; in particular, Spalding (1959) used

$$G(B) = B^{-1} \ln(1+B), \quad [7]$$

whereas Eisenklam *et al.* (1967) preferred

$$G(B) = (1+B)^{-1}, \quad [8]$$

although the validity of neither of them has been unassailably established (Natarajan 1973). However, since B is a constant during droplet motion, we shall leave $G(B)$ as an unspecified constant function in the formulation. It may also be noted that $G(B) \approx 1$ in the limit of $B \ll 1$ for slow vaporization.

The functions $K(Sc)$ and $H(Re, Sc)$ are inserted in [6] so that for a given Sc and for $G(B) = 1$, a value for $K(Sc)$ is determined that enables the resulting C_D to closely correlate the standard experimental drag curve for solid spheres. The function

$$K(Sc) = 23Sc^{-0.14} \quad [9]$$

was found to meet the above purpose for $0.5 < Sc < 10$ and $Re < 200$ (figure 1). It is seldom necessary to investigate the behavior of the system for larger Reynolds numbers, since for most liquids which are not too viscous the droplets would then become unstable and tend to break up. Figure 1 shows that the present drag-coefficient expression satisfactorily correlates the standard drag curve for the range of Sc illustrated, implying that any uncertainty in the selection of Sc will not significantly affect the value of the calculated drag-coefficient. As Re approaches 0, the present correlation fails to asymptote to the Stokes expression $C_D = 24/Re$. The difference, however, is small and hardly important since under such situations the droplet conforms to the gas motion almost instantly. Finally, it should be emphasized that the primary reason for adopting the function $H(Sc, Re)$, in preference to other well-known correction factors to the Stokes drag coefficient, is because its presence greatly facilitates the following mathematical analysis, such that further physical insight can be gained in delineating the various factors governing the droplet motion.

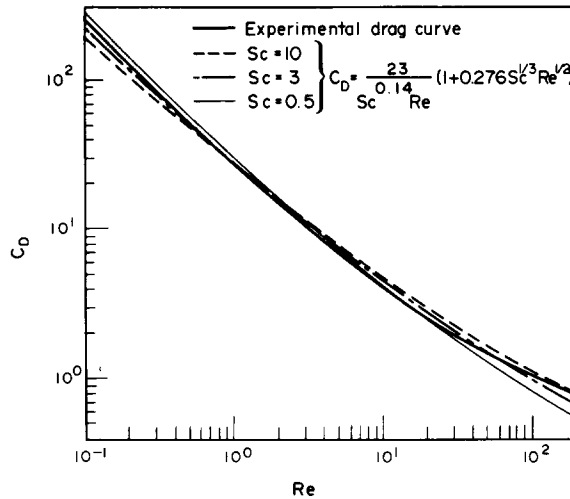


Figure 1. Comparison between experimental drag-coefficient and the present correlation.

3. SOLUTIONS AND DISCUSSIONS

Dividing [4] and [5] by [1], the nonlinear term $H(Re, Sc)$ is eliminated and the following linear equations are obtained for u_p and v_p ,

$$du_p/d\hat{D}^2 = -(\alpha/2)(u_G - u_p)/\hat{D}^2, \quad [10]$$

$$dv_p/d\hat{D}^2 = (\alpha/2)v_p/\hat{D}^2, \quad [11]$$

where $\alpha = (3/16)\mu_G K(Sc)G(B)/M$. Integrating [10] and [11], with the initial conditions that $u_p = u_{p0}$ and $v_p = v_{p0}$ at $\hat{D} = 1$, yields

$$u_p = u_G + (u_{p0} - u_G)\hat{D}^\alpha \quad [12]$$

and

$$v_p = v_{p0}\hat{D}^\alpha. \quad [13]$$

The function $H(Re, Sc)$ is now expressed in terms of \hat{D} as

$$H(Re, Sc) = 1 + \beta\hat{D}^{(1+\alpha)/2}, \quad [14]$$

where $\beta = 0.276Sc^{1/3}Re_o^{1/2}$.

Equations [12] and [13] indicate that since $u_p \rightarrow u_G$, $v_p \rightarrow 0$ only when $\hat{D} \rightarrow 0$, the droplet velocity is never in phase with the gas velocity. Therefore the penetration depth y_{\max} attained by the droplet, when $v_p = 0$, corresponds to the state of complete vaporization.

An important parameter, α , is also identified. It represents the ratio of the effects of drag to the rate of evaporation; viz., $[\mu_G G(B)]/M$. Hence for $\alpha \ll 1$ the droplet vaporizes so fast that it retains its initial velocity, the drag force having little influence on its motion. Only during the last stage of its lifetime, when the droplet size becomes very small, will the drag force dominate and the droplet quickly conform to the gas motion. For $\alpha \gg 1$ the reverse is true. The droplet quickly loses its inertia after injection and is almost in phase with the gas motion before much of its initial mass is vaporized. The rest of its lifetime is spent in vaporization and in a final, slight, adjustment to follow the gas motion. Equations [12] and [13] illustrate the above behavior by showing that for $\alpha \ll 1$, $\hat{D}^\alpha \rightarrow 1$ for $\hat{D} \ll 1$ such that $u_p \rightarrow u_{p0}$ and $v_p \rightarrow v_{p0}$; whereas for $\alpha \gg 1$, $\hat{D}^\alpha \rightarrow 0$ for $(1 - \hat{D}) \ll 1$ such that $u_p \rightarrow u_G$ and $v_p \rightarrow 0$.

The above discussion also implies that for a slowly vaporizing droplet the analysis can be greatly simplified by first suppressing the vaporization process until the droplet velocity is

completely in phase with the gas, then allowing the droplet to vaporize in effectively a stagnant atmosphere.

The droplet size can be related to the y -coordinate by using [1] and [13] and the identity

$$d/dt = v_p d/dy, \quad [15]$$

leading to

$$y = \frac{v_{po}}{(2 + \alpha)k} \Phi_y(\lambda), \quad [16]$$

where we have defined

$$\Phi_y(\lambda) = \nu_1 \int_{\lambda}^1 \frac{\lambda'^{(3+\alpha)/(1+\alpha)}}{1 + \beta\lambda'} d\lambda' \quad [17]$$

with

$$\lambda = \hat{D}^{(1+\alpha)/2}$$

and

$$\nu_1 = 2(2 + \alpha)/(1 + \alpha).$$

It may be noted that $\Phi_y(0) = 1$ for $\beta = 0$. The penetration depth is now given by

$$y_{\max} = \frac{v_{po}}{(2 + \alpha)k} \Phi_y(0). \quad [18]$$

A similar derivation can be extended to the x -component of motion, giving

$$x = \frac{(u_{po} - u_G)}{(2 + \alpha)k} \Phi_x(\lambda), \quad [19]$$

where

$$\Phi_x(\lambda) = \frac{\nu_1 u_G}{(u_{po} - u_G)} \int_{\lambda}^1 \frac{\lambda'^{(3-\alpha)/(1+\alpha)}}{1 + \beta\lambda'} d\lambda' + \Phi_y(\lambda). \quad [20]$$

The chamber length for complete droplet vaporization is then

$$x_{\max} = \frac{(u_{po} - u_G)}{(2 + \alpha)k} \Phi_x(0). \quad [21]$$

When the initial Reynolds number, Re_o , is not too large so that $\beta < 1$, then $\Phi_x(\lambda)$ and $\Phi_y(\lambda)$ can be expressed in terms of the Gauss hypergeometric series (Gradshteyn 1965), giving

$$\Phi_y(\lambda) = F(1, \nu_1; 1 + \nu_1; -\beta) - \lambda^{\nu_1} F(1, \nu_1; 1 + \nu_1; -\beta\lambda), \quad [22]$$

$$\Phi_x(\lambda) - \Phi_y(\lambda) = \frac{\nu_1 u_G}{(u_{po} - u_G) \nu_2} \{F(1, \nu_2; 1 + \nu_2; -\beta) - \lambda^{\nu_2} F(1, \nu_2; 1 + \nu_2; -\beta\lambda)\} \quad [23]$$

where $\nu_2 = 4/(1 + \alpha)$.

It is of interest to compare the present penetration depth with that for a solid particle obeying Stokes law, which is given by

$$(y_{\max})_{\text{Stokes}} = v_{po} \rho_p D_o^2 / (18\mu_G). \quad [24]$$

The ratio of the depths, $R = (y_{\max})_{\text{Stokes}}/y_{\max}$, is

$$R = \left\{ \frac{K(Sc)}{24\Phi_y(0)} \right\} \left\{ G(B) + \frac{32 \ln(1+B)}{3ScK(Sc)} \right\}. \quad [25]$$

The deviation of the above ratio from unity is due to three factors. The first bracketed term accounts for the adjustments made in the Stokes law to conform to the standard drag curve for solid spheres. For $\beta = 0$ and by setting $K(Sc) = 24$, this term is 1. The second bracketed term accounts for the effects due to vaporization. The function $G(B)$ is the correction for the drag coefficient due to evaporation, whereas the second term accounts for the continuous increase in the drag force, per unit mass, due to the diminishment of the droplet size. For solid spheres or for very slowly vaporizing droplets $B \approx 0$ and $G(B) \approx 1$, hence the second bracketed term is approx. 1.

As an illustration, the ratio R is calculated for a heptane droplet with $D_o = 10^{-2}$ cm and $v_{po} = 10^2$ cm/sec, injected perpendicularly into a 1 atm air stream with $Sc = 0.7$. For $T_G = 300^\circ\text{K}$ and 500°K , B is found (Law 1975*b*) to be 0.143 and 1.33; $[32 \ln(1+B)]/[3ScK(Sc)]$ is 0.0495 and 0.312; $G(B)$, using [7], is 0.94 and 0.635; $\Phi_y(0)$ is 0.698 and 0.734; hence R is 1.43 and 1.30, respectively. The above figures confirm that at $T_G = 300^\circ\text{K}$, since vaporization is very slow, $G(B) = 0.94 \approx 1$ and the droplet can be effectively treated as a solid sphere in computing its trajectory. This assumption, however, fails at $T_G = 500^\circ\text{K}$. It may also be noted that at $T_G = 500^\circ\text{K}$, whereas the rapid diminishment in the droplet size tends to significantly increase the drag force per unit mass, the drag reduction due to the enhanced mass-transfer rate at the droplet surface more than compensates for this increase so that the penetration depth is larger than for the case at $T_G = 300^\circ\text{K}$.

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